W02 - Probability and parameter inference

Another example:

The effectiveness (f) of a new mRNA vaccine

N subject
n disease free after 2 mo $\rightarrow$ infer effectiveness $0 < f \leq 1$

A Bernoulli process

an event happens w/ prob p.

* $N \rightarrow n$ - Binomial distribution.

$$P(n|N,p) = \binom{N}{n} p^n (1-p)^{N-n}$$

$$\langle n \rangle = Np$$

$$\sigma^2 = Np(1-p) \qquad n = 0, \ldots, N$$

* event happens n times and no more

$$P(n|p) = (1-p)p^n$$

$$n = 0, \ldots \infty \qquad \langle n \rangle = p/1-p \qquad \sigma^2 = \frac{p}{(1-p)^2}$$

data = n/N are disease free

$$P(\text{data} \mid f, N) = \frac{N!}{n!\,(N-n)!} f^n (1-f)^{N-n}$$

$$P(f) = 1$$

$$P(f \mid n\, N) = \frac{P(n \mid N, f) \cdot P(f)}{P(n \mid N)}$$

$$= \frac{P(n \mid N, f) \cdot P(f)}{\int_0^1 P(n \mid N, g)\, P(g)\, dg}$$

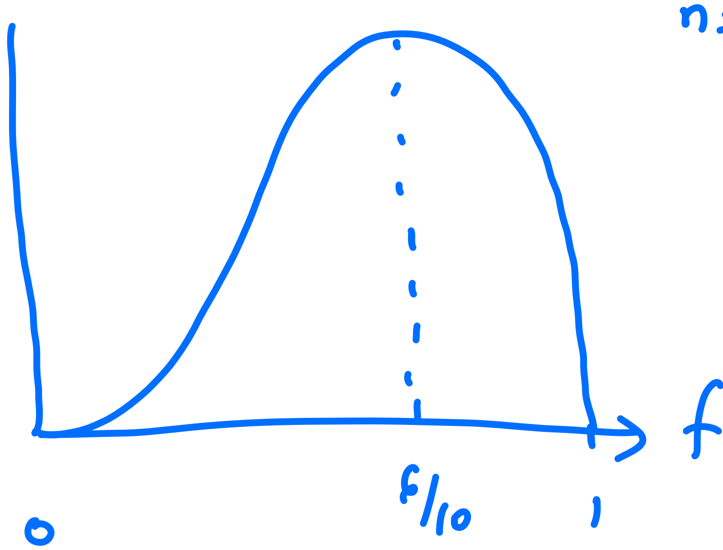
$$= \frac{f^n (1-f)^{N-n}}{\int_0^1 g^n (1-g)^{N-n}\, dg}$$

Gamma integral

$$\int_0^1 g^n (1-g)^{N-n}\, dg = \frac{n!\,(N-n)!}{(N+1)!}$$

$$P(f|n,N) = \frac{(N+1)!}{n!\,(N-n)!} f^n (1-f)^{N-n}$$

$N = 10$
$n = 6$



Compare

$$P(n \mid N f) = \frac{N!}{n!\,(N-n)!} f^n (1-f)^{N-n}$$

$$\sum_{n=0}^{N} P(n \mid N f) = 1 \quad \forall f$$

$$P(f \mid n N) = \frac{(N+1)!}{n!\,(N-n)!} f^n (1-f)^{N-n}$$

$$\int_0^1 P(f \mid n N)\, df = 1$$

$P(\text{data} \mid f) \longrightarrow P(f \mid \text{data})$

$P(n \mid N f)$ $\qquad\qquad$ $P(f \mid n N)$

Using posterior probabilities to make further inferences

P(next subject is plague free $| n, N$) =

$$= \int_0^1 f \; P(f|nN) \, df$$

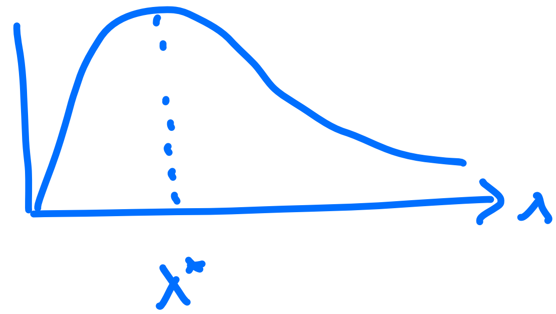
$$= \frac{(N+1)!}{n!\,(N-n)!} \int_0^1 f^{n+1} (1-f)^{N-n} \, df$$

$$= \frac{(N+1)!}{n!\,(N-n)!} \; \frac{(n+1)!\,(N-n)!}{(N+2)!} = \frac{n+1}{N+2}$$

$$\langle f \rangle = \frac{n}{N} \rightarrow \frac{n+1}{N+2}$$

Parameters best estimates and confidence intervals

$P(\lambda|D)$



$L(\lambda) = \log P(\lambda|D)$

A Taylor expansion around λ^*

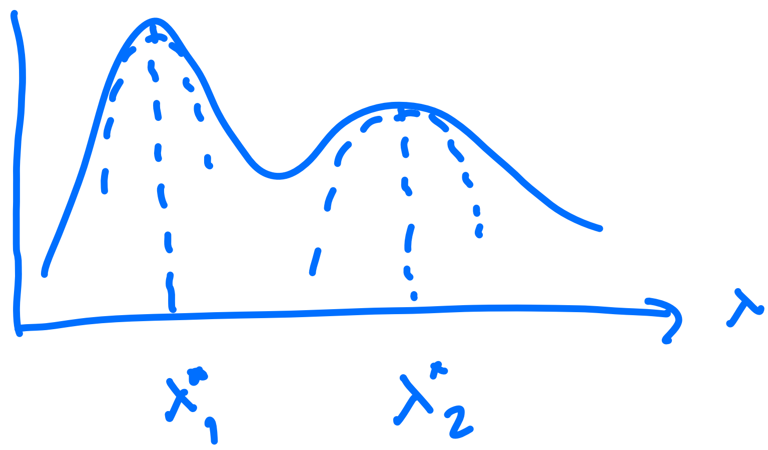
$$L(\lambda) \simeq L(\lambda^*) + \cancelto{0}{\frac{\delta L}{\delta \lambda}\Big|_{\lambda^*}(\lambda-\lambda^*)} \qquad \text{''} \quad \frac{\delta L}{\delta \lambda}\Big|_{\lambda^*} = 0$$

$$+ \frac{1}{2}\frac{\delta^2 L}{\delta \lambda^2}\Big|_{\lambda^*}(\lambda-\lambda^*)^2 + O(\lambda-\lambda^*)^3$$

$$P(\lambda|D) \simeq e^{L} \simeq e^{L(\lambda^*) + \frac{1}{2}\frac{\delta^2 L}{\delta\lambda^2}\big|_{\lambda^*}(\lambda-\lambda^*)^2}$$

a Normal distribution around λ^*

$$-\frac{\delta^2 L}{\delta \lambda^2}\Big|_{\lambda^*} = \frac{1}{\sigma^2}$$



$$\frac{1}{\sigma_1^2} = -\frac{\delta^2 L}{\delta \lambda^2}\Big|_{\lambda_1^*}, \quad \frac{1}{\sigma_2^2} = -\frac{\delta^2 L}{\delta \lambda^2}\Big|_{\lambda_2^*}$$

the laplace approximation

MacKay chapter 27, p341

For the bacterial wait times

$$P(\lambda \mid \text{data}) \propto \frac{e^{-\mu N/\lambda}}{Z^N(\lambda)}$$

$$\hat{t} = \frac{1}{N}\sum_i t_i \quad (\text{the data})$$

$$Z^N(\lambda) \simeq \lambda^N$$

$$\log P(\lambda \mid \text{data}) \propto -\frac{\hat{t}N}{\lambda} - \log \lambda^N$$

$$= -\frac{\hat{t}N}{\lambda} - N \log \lambda$$

$$\frac{\delta L}{\delta \lambda} = \frac{\hat{t}N}{\lambda^2} - \frac{N}{\lambda} \Bigg| = 0 \qquad \frac{\hat{t}}{\lambda^*} = 1 \; \Bigg\{ \boxed{\lambda^* = \hat{t}}$$

$$\frac{\delta^2 L}{\delta\lambda^2} = -2\frac{\hat{t}N}{\lambda^3} + \frac{N}{\lambda^2} = \frac{N}{\lambda^2}\left(1 - \frac{2\hat{t}}{\lambda}\right)$$

$$\left.\frac{\delta^2 L}{\delta\lambda^2}\right|_{\lambda=\hat{t}} = \frac{N}{\hat{t}}(1-2) = -\frac{N}{\hat{t}}$$

$$\sigma^2 = \frac{\hat{t}^2}{N} \qquad \mu = \frac{\sum_i t_i}{N} = \hat{t}$$

$$\underbrace{\sigma \propto \frac{1}{\sqrt{N}}}$$

the error in the derivation of the parameters is always proportional to the inverse of the square root of the data $1/\sqrt{N}$