W03 - Model comparison and hypothesis testing

$P(D|H)$

$\downarrow$

$P(H|D)$

D = data

H = hypothesis

Compare

$$P(H_1|D) = \frac{P(D|H_1)\cdot P(H_1)}{P(D)}$$

$$P(H_2|D) = \frac{P(D|H_2)\cdot P(H_2)}{P(D)}$$

$$\frac{P(H_1|D)}{P(H_2|D)} = \frac{P(D|H_1)\cdot P(H_1)}{P(D|H_2)\cdot P(H_2)}$$

A Practical case   Vaccine trials

- 40
  - $N_V = 30$ — V
    - n_V: 1 — n_V contract disease
    - 29 — $(N_V - n_V)$ not
  - $N_c = 10$ — c
    - n_c: 3 — n_c contract disease
    - 7

Q: is V better than C ?

f_V = probability of contracting disease in V group

f_c = " " " in C group

$H_1 = [V \text{ is better than } C] = f_V < f_C$

$H_0 = [V \cong C] \qquad = f_V = f_C$

- → classical χ^2 - test
- Bayesian Dnf $\dfrac{P(H_1|D)}{P(H_2|H)}$

χ^2 test

* Observed values $\{O_i\}_{i=1}^{N}$

* expected values $\{E_i\}_{i=1}^{N}$
  under a Null hypothesis

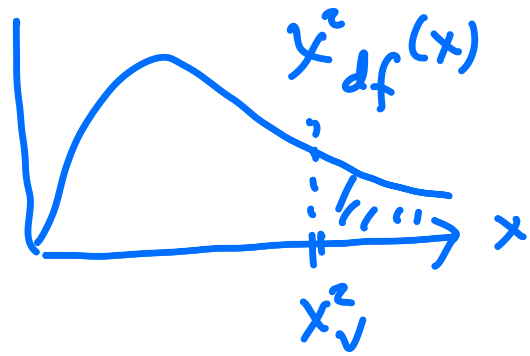
i) Calculates

$$\chi_\nu^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

ii) Assumes χ_ν^2 follows a χ^2 distribution

df = degrees of freedom

= N − # constraints



$$\text{p-val}(\chi_\nu^2) = P(x \geq \chi_\nu^2) = 1 - CDF(\chi_\nu^2)$$

χ^2 test for our example

			O	E
V	$\frac{N_v}{30}$	n_v	1	$N_v \cdot f_0 = \frac{30}{10} = 3$
			29	$N_v \cdot (1-f_0) = 30 \frac{9}{10} = 21$
C	$\frac{N_c}{10}$	n_c	3	$N_c \cdot f_0 = \frac{10}{10} = 1$
			7	$N_c \cdot (1-f_0) = 10 \cdot \frac{9}{10} = 9$

Null Hypothesis = $H_0 = (f_v = f_c)$

f_{H_0} = prob of getting disease if $f_v = f_c$

$$= \frac{1+3}{40} = \frac{1}{10}$$

$$\chi^2_v = \frac{(1-3)^2}{3} + \frac{(29-21)^2}{21} + \frac{(3-1)^2}{1} + \frac{(7-9)^2}{9}$$

$$= 5.926$$

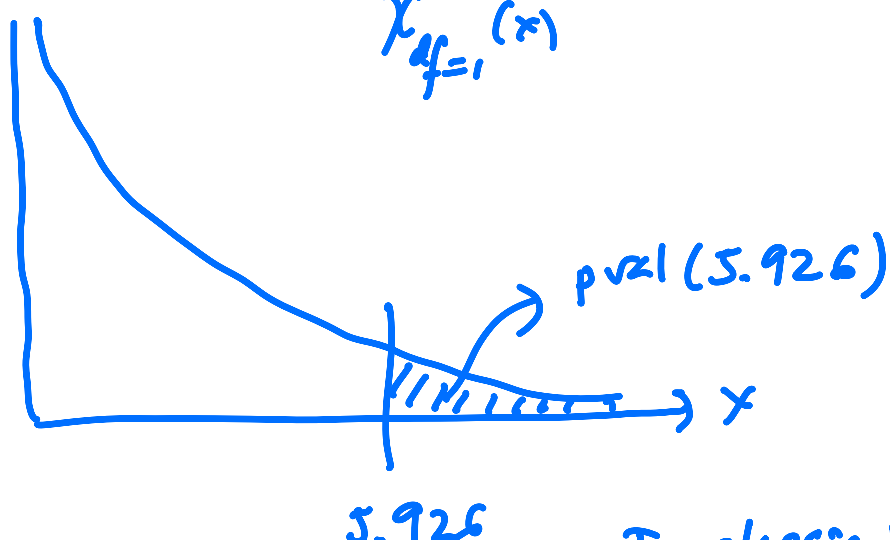
$$df = 4 - 3 = 1$$

$N_c = 10$

$N_v = 30$

$f_0 = \frac{1}{10}$

$\chi^2_{df=1}(x)$



$$\boxed{\text{pval}(5.926) = 0.015}$$

In classical stats if pval < 0.05 can reject data following H_0 $(f_c = f_v)$

Notes abt χ^2 test

(1) p-val $= 0.015$ does not mean

that H_1 ($f_V < f_C$) has a prob of 98.5%
$V > C$

(2) what it means is,

If we were to repeat the same experiment ($N_V = 30$, $N_C = 10$) many times and ($V = C$), we would expect χ^2 to have value 5.926 or higher $\simeq$ 1.5% of times

(3) χ^2 does use H_0 ($f_0 = \frac{1+3}{40}$)

does not use H_1
[not $n_V = 1$, $n_C = 3$]

How does χ^2 test work?

Given df random variables $X_1 \dots X_{df}$ such that each follows a $N(\mu=0, \sigma=1)$ distribution then

$$X_1 + \dots + X_{df} \sim \chi^2_{df}$$

assumes: $$\frac{(O_i - E_i)^2}{E_i} \sim N(0,1)$$

Can we test this statement?

Which χ^2?

$$\text{pearson } \chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

$$\text{yates } \chi^2 = \sum_i \frac{(|O_i - E_i| - 0.05)^2}{E_i}$$

$\chi^2_{yates} = 3.34 \rightarrow$ pval $= 0.7$ not reject null Hypothesis?

Bayesian Inference

$$\frac{P(H_{v \neq c} | D)}{P(H_{v=c} | D)}$$

"Bayes or χ^2, or does it matter?" [Mackay]

$$D = (\underbrace{n_v, N_v}_{D_v}, \underbrace{n_c\, N_c}_{D_c}) \quad \text{data.}$$

$$P(D_v | f_v) = \frac{N_v!}{n_v!\,(N_v - n_v)!} f_v^{n_v} (1 - f_v)^{N_v - n_v}$$

a binomial distribution.

- Assume independence

$$P(D | f_v\, f_c) = P(D_v | f_v) \cdot P(D_c | f_c)$$

- priors too, and assume uniform

$$P(f_v\, f_c) = P(f_v) \cdot P(f_c) = 1$$

the posterior of parameters

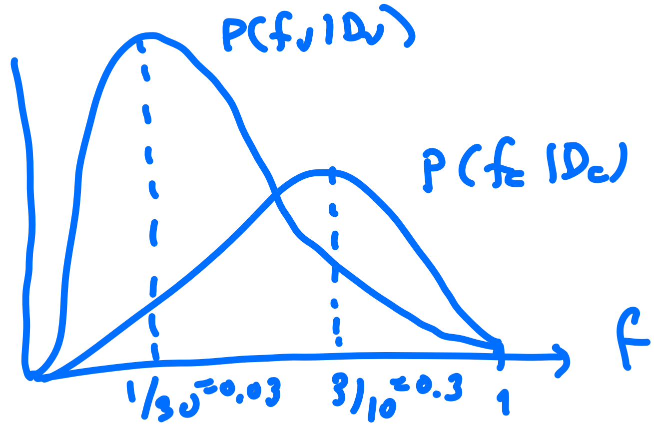
$$P(f_v f_c | D) = \frac{P(D_v|f_v) \cdot P(f_v) \; P(D_c|f_c) \cdot P(f_c)}{P(D_v) \cdot P(D_v)}$$

$$= P(f_v | D_v) \cdot P(f_c | D_c)$$

WØ2

$$P(f_v | D_v) = \frac{(N_v+1)!}{n_v!(N_v-n_v)!} f_v^{n_v} (1-f_v)^{N_v-n_v}$$

$$P(f_c | D_c) = \frac{(N_c+1)!}{n_c!(N_c-n_c)!} f_c^{n_c} (1-f_c)^{N_c-n_c}$$



Bayesian hypothesis comparison

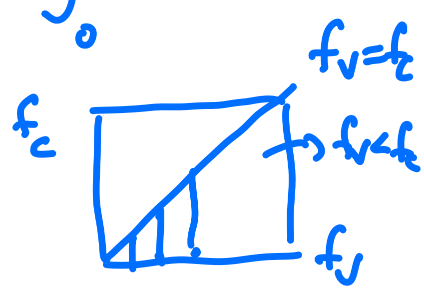
$H_1 : f_v < f_c$

$H_0 : f_v = f_c$

$$P(H_1 | D) = P(f_v < f_c | D)$$

$$= \int_0^1 df_c \int_0^{f_c} df_v \, P(f_v | D_v) \, P(f_c | D_c)$$

$$= \frac{(N_v+1)!}{n_v!\,(N_v-n_v)!} \cdot \frac{(N_c+1)!}{n_c!\,(N_c-n_c)!} \int_0^1 df_c \, f_c^{n_c} (1-f_c)^{N_c-n_c} \int_0^{f_c} df_v \, f_v^{n_v} (1-f_v)^{N_v-n_v}$$



$$= 0.98$$

$N_v = 30,\ n_v = 1$

$N_c = 10 \ \ n_c = 3$

$$P(H_0 \mid D) = P(f_c = f_v \mid D)$$

$$= \int_0^1 df \; P(f_c = f \mid D_c) \cdot P(f_v = f \mid D_v)$$

$$= \frac{(N_v+1)!}{n_v!\,(N_v-n_v)!} \cdot \frac{(N_c+1)!}{n_c!\,(N_c-n_c)!} \times$$

$$\times \int_0^1 df \; f^{n_c+n_v} (1-f)^{N_c+N_v-(n_c+n_v)}$$

$$= 0.33$$

$$\boxed{\frac{P(H_1 \mid D)}{P(H_0 \mid D)} = \frac{0.98}{0.33} = 3.01}$$

"This is the factor by which the data should modify your beliefs about the two Hypotheses"

(Mackay)

the effectiveness

$$\boxed{H_2 : f_V < \alpha f_C \qquad \alpha < 1}$$

$1 - \alpha \equiv$ effectiveness

$$P(f_V < \alpha f_C | D) =$$

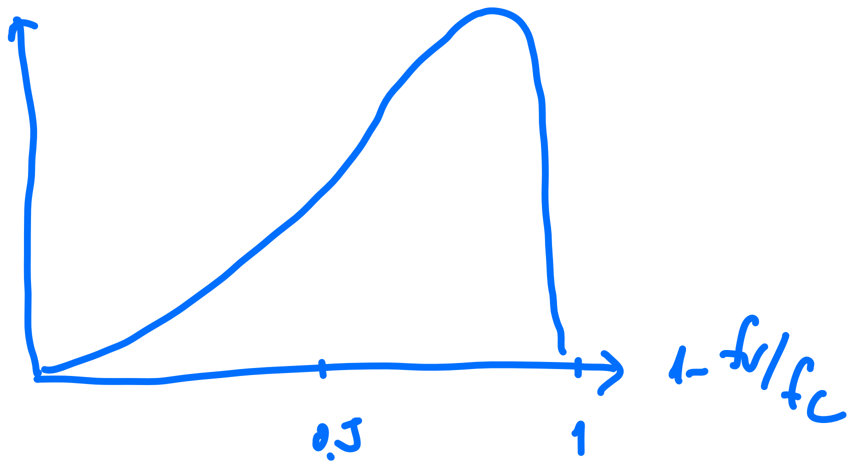
$$= \int_0^1 df_C \int_0^{\alpha f_C} df_V \; P(f_C | D_C) \cdot P(f_V | D_V)$$

$$= \frac{(N_V+1)!}{n_V!\,(N_V-n_V)!} \frac{(N_C+1)!}{n_C!\,(N_C-n_C)!} \,\cdot$$

$$\cdot \int_0^1 df_C \; f_C^{n_C} (1-f_C)^{N_C-n_C} \int_0^{\alpha f_C} df_V \; f_V^{n_V} (1-f_V)^{N_V-n_V}$$

$N_v = 30$ $N_c = 10$

$n_v = 1$ $n_c = 3$



Moderna data Nov 2020

$N_v = N_c = 15{,}000$

$n_c = 5$

$n_v = 90$